The Open University

M332/K

Third Level Course Examination 1989

Complex Analysis

Friday, 20 October, 1989 **2.30 pm–5.30 pm**

Time allowed: 3 hours

There are **THREE** parts to this paper.

In Part I, you should attempt as many questions as you can. You should attempt not more that **TWO** questions in Part II and not more than **ONE** question in Part III.

You are advised to spend approximately $1\frac{1}{2}$ **hours** on Part I (50% of total marks), 1 **hour** on Part II (34% of total marks), and $\frac{1}{2}$ **hour** on Part III (16% of total marks).

At the end of the examination:

Check that you have written your name, personal identifier and examination number on **each** answer book used. **Failure to do so will mean that your work cannot be identified.**

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PART I

You should attempt as many questions as you can in this part of the examination.

The questions in this part carry 50% of the total marks so you should spend about $1\frac{1}{2}$ **hours** *in answering them.*

You may answer the questions in any order. Write your answers in the answer books provided, beginning each question on a new page.

Show all your working.

Question 1

For $z_0 = 2 + 2\sqrt{3}i$, write down

(i) the imaginary part of z_0;

(ii) a polar form of z_0;

(iii) $\dfrac{1}{z_0^2}$ in a polar form;

(iv) the square roots of z_0 in the form $a + ib$ where a and b are real numbers;

(v) a real quadratic equation (i.e. one with real coefficients) which has z_0 as one of its roots. [5]

Question 2

Make separate sketches of

$$A = \{z : 1 < |i - z| < 3\}, B = \{z : z \neq 0, \pi/3 < \operatorname{Arg} z < 2\pi/3\}.$$

For each of the sets A and B state whether or not it is

(i) connected, (ii) simply-connected, (iii) a region. [5]

Question 3

(i) Sketch, or otherwise describe, the inverse image under the function $z \longrightarrow z^2$ of the upper half-plane $\{z : \operatorname{Im} z \geq 0\}$. [2]

(ii) Sketch, or otherwise describe, the image of the rectangle $\{z : 0 \leq \operatorname{Re} z \leq 1,\ 0 \leq \operatorname{Im} z \leq \pi/2\}$ under the function $z \longrightarrow e^z$. [3]

Question 4

(i) Write down the values of

(a) $\operatorname{Log}(i)$; (b) $\operatorname{Log}(1 + i)$.

(ii) What are the values of i^i? Indicate which is the principal value.

(iii) Determine the domain of $\operatorname{Log}(1 - iz)$ giving your answer in set notation or as a clear sketch. [5]

Question 5

(i) Evaluate

$$\int_\gamma \operatorname{Im} z \, dz$$

where γ is the arc defined by

$$\gamma(t) = 1 + t(i-1), \quad t \in [0,1]$$

from 1 to i. [2]

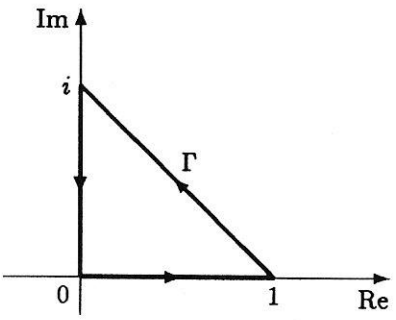
(ii) Evaluate

(a) $\int_{[i,0]} \operatorname{Im} z \, dz$ and (b) $\int_{[0,1]} \operatorname{Im} z \, dz$,

and hence, or otherwise, evaluate

$$\int_\Gamma \operatorname{Im} z \, dz$$

where Γ is the triangular contour shown below.



[3]

Question 6

(i) State Cauchy's Formula for the nth derivative. [2]

(ii) Evaluate the following integrals, where C is the circle with centre 2 and radius 2:

(a) $\int_C \frac{\sin z}{(z-\pi)^2} \, dz$; (b) $\int_C \frac{\sin z}{(z+\pi)^2} \, dz$. [3]

Question 7

(i) Write $\frac{1}{(z-1)(z-3)}$ in partial fraction form. [1]

(ii) Hence, or otherwise, find the terms in z^{-2}, z^{-1}, z and the constant term, if any, of the Laurent series for

$$f : z \longrightarrow \frac{1}{(z-1)(z-3)}$$

which is convergent in the annulus $\{z : 1 < |z| < 3\}$. [4]

Question 8

(i) Classify the indicated singularity of each of the following functions as a removable singularity, a pole (stating order) or an essential singularity:

(a) $f(z) = \dfrac{\sin z}{z^4}$ at 0;

(b) $g(z) = \dfrac{z}{\sin z}$ at π;

(c) $h(z) = (z-1)\sin\left(\dfrac{1}{(z-1)}\right)$ at 1.

[No explanations required.]

(ii) Calculate the residue at each singularity considered in part (i). [6]

Question 9

Evaluate

$$\int_C \frac{e^z}{z(z^2+4)}\,dz,$$

where $C = \{z : |z| = 3\}$. [4]

Question 10

(i) What conditions must be satisfied by two function elements (f_1, R_1) and (f_2, R_2) for (f_2, R_2) to be a direct analytic continuation of (f_1, R_1)? [1]

(ii) If $f_1(z) = \sum_{n=0}^{\infty}(1-z)^n$ on $R_1 = \{z : |1-z| < 1\}$ and $f_3(z) = -\sum_{n=0}^{\infty}(1+z)^n$ on $R_3 = \{z : |1+z| < 1\}$, show by finding a suitable function element (f_2, R_2) that f_1 and f_3 are branches of the same complete analytic function (analytic multifunction). [4]

PART II

You should attempt not more than **TWO** *questions from this part of the examination.*

Each question in this part carries 17% of the total marks and so you should spend about 1 hour in answering Part II as a whole.

You may answer the questions in any order. Write your answers in the answer books provided, beginning each question on a new page.

Show all your working.

Question 11

(i) (a) State both forms of the triangle inequality for complex numbers. [1]

(b) State and prove the Estimation Theorem for contour integrals, stating clearly any auxiliary results that you use. [7]

(ii) Determine an upper bound for

$$\left| \int_\Gamma \frac{z + \operatorname{Log} z}{e^z + 3}\, dz \right|$$

where Γ is the semi-circular contour $\{z : z = e^{i\theta},\ -\pi/2 < \theta < \pi/2\}$. [9]

Question 12

(i) (a) Write down the Cauchy-Riemann equations satisfied by a function

$$f(x + iy) = U(x, y) + iV(x, y),$$

where U and V are real functions, which is differentiable at a point $z_0 = x_0 + iy_0$. [2]

(b) Determine whether or not the Cauchy-Riemann equations hold for the function

$$f(x + iy) = x^2 + y^2 - y + ix$$

at either or both of the points 0 and $1 + i$. [5]

(c) Determine if f defined in (b) above is differentiable at either or both of the points 0, $1 + i$ and if so write down its derivative at this/these points. [3]

(ii) (a) Write down the definition of differentiability at a point z_0 for a complex function f, and use it to show that

$$g : z \longrightarrow \operatorname{Re} z$$

is not differentiable anywhere. [5]

(b) Decide whether the following statement is true or false. If true prove it: if false give a counterexample to demonstrate this.

Statement: If complex functions h_1 and h_2 are not differentiable anywhere then $h = h_1 + ih_2$ is not differentiable anywhere. [2]

Question 13

Using any valid method, evaluate each of the following integrals, giving your answer in as simplified a form as possible. Quote any standard results that you use, checking required conditions as necessary.

Γ is the upper half of the unit circle centred on 0 from 1 to -1,
i.e. $\{z : |z| = 1,\ \mathrm{Im}\, z \geq 0\}$, and C is the circle $\{z : |z| = 2\}$.

(i) $\displaystyle\int_\Gamma \frac{1}{z}\, dz;$ [3]

(ii) $\displaystyle\int_\Gamma |z|\, dz;$ [3]

(iii) $\displaystyle\int_C \frac{z}{\sin z}\, dz;$ [2]

(iv) $\displaystyle\int_C \frac{1}{1+z^3}\, dz;$ [5]

(v) $\displaystyle\int_C \frac{\cosh z}{(z-\log 2)^6}\, dz.$ [4]

Question 14

(i) (a) State the Maximum Principle. [2]

(b) Find the maximum value of the function

$$z \longrightarrow \left|\frac{z^2-3}{z^2+3}\right|$$

on the set $\{z : |z| \leq 1\}$, and find the point or points where this maximum is attained. [8]

(ii) In this part of the question you may use any theorems of the course as long as you quote them, either by name or explicitly, and check that they are indeed applicable.

(a) Let D_a be an open disc with centre a. Prove that sin is bounded in D_a. [2]

(b) Prove that sin is not bounded in $\mathbf{C}$. [2]

(c) Deduce from parts (a) and (b) and the definition of compactness that $\mathbf{C}$ is not compact by showing that if it were then sin would be bounded in $\mathbf{C}$. [3]

Question 15

(i) Evaluate the following integrals. If you use any standard theorems you must check that the conditions required by the theorem hold:

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{t \sin 2t}{t^2+1}\, dt;$

(b) $\displaystyle\int_0^{\infty} \frac{1}{t^2-4}\, dt.$ [5,6]

(ii) Find a function f such that

$$\int_{-\pi}^{\pi} \frac{1}{5+4\cos\theta}\, d\theta = \int_C f(z)\, dz$$

where $C = \{z : |z| = 1\}$, and hence find the value of the first integral. [6]

Question 16

(i) State, without proof, the Principle of the Argument. [2]

(ii) State and prove Rouché's Theorem.
(You may assume that if ϕ is analytic on a region R and Γ is a closed contour in R such that $|\phi| < 1$ on Γ then $\displaystyle\int_\Gamma \frac{\phi'}{1+\phi} = 0$.) [7]

(iii) Use Rouché's Theorem to show that

$$h(z) = 6z^2 + 2 + e^{iz}$$

has two zeros in the disc $\{z : |z| < 2\}$ and that exactly one of these zeros is in the upper half-plane $\{z : \operatorname{Im} z > 0\}$. [8]

PART III

You should attempt **ONE** *question from this part of the examination.*

This part carries 16% of the total marks and so you should spend about $\frac{1}{2}$ **hour** *in answering it.*

Write your answer in one of the answer books provided and show all your working.

Question 17

(i) Define the terms conformal mapping, bilinear transformation. [2]

(ii) This section leads to finding an analytic equivalence of the region $A = \{z : |z - i| < 1, |z - 1| < 1\}$, formed by the intersection of two discs, and the open unit disc $D = \{z : |z| < 1\}$.

(a) Sketch, on separate diagrams, the sets A, B, C and D, where

$$B = \{z : \pi/4 < \operatorname{Arg} z < 3\pi/4\} \text{ and } C = \{z : \operatorname{Re} z < 0\}.$$ [2]

(b) Show that the bilinear transformation

$$f : z \longrightarrow \frac{z - (1 + i)}{iz}$$

maps A onto B. [5]

(c) State an analytic transformation g which maps B onto C. [1]

(d) Determine a bilinear transformation which maps C onto D. [4]

(e) Hence write down, without simplifying it, a formula for an analytic equivalence of A and D.
Justify that your answer is indeed an analytic equivalence. [2]

Question 18

(i) Write down the definition of the Laplace transform F of a function f and use it to show

(a) $\mathcal{L}[f''](p) = p^2F(p) - pf(0) - f'(0)$; [3]

(b) $\mathcal{L}[\sin](p) = \dfrac{1}{p^2 + 1}$. [3]

(ii) Let f be the real function which satisfies the differential equation

$$f''(t) + f(t) = \sin t,$$

where $f'(0) = f(0) = 0$.

(a) Determine the Laplace transform of f. [2]

(b) Verify that you may use the Inversion Formula to evaluate the function f. [2]

(c) Hence, using the Inversion Formula, show that

$$f(t) = \frac{1}{2}(\sin t - t \cos t).$$ [6]

[END OF QUESTION PAPER]

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

M332 Complex Analysis

M332 Solutions to 1989 Examination Paper

COMPLEX ANALYSIS

Solutions to the 1989 Examination Paper

Before reading the enclosed solutions you should attempt the questions yourself. The solutions given are minimal in the sense that they give the least amount of detail that would be acceptable for full marks. We would expect most students to put in a lot more steps and use more diagrams. You will probably find the solutions hard to follow if you do not do your own working.

Do read the comments. They give useful hints to solving the problems and to what the examiner is looking for. For example, you will see that when we ask for the statement of a theorem we are not looking for a verbatim quote but for an understanding of what the theorem means and, in particular, the conditions under which it holds. Notice also when applying a theorem that you should check that its requirements hold.

Question 1

(i) $2\sqrt{3}$

(ii) $4e^{i\pi/3}$

(iii) $\frac{1}{16}e^{-i2\pi/3}$

(iv) $\pm 2e^{i\pi/6} = \pm(\sqrt{3}+i)$.

(v) The two roots of the real quadratic equation must be complex conjugates, so a suitable equation is

$$(z-z_0)(z-\overline{z}_0)=0$$

i.e. $(z-2-2\sqrt{3}i)(z-2+2\sqrt{3}i)=0$

i.e. $z^2-4z+16=0$.

Comment *The answer in* (ii) *is in acceptable polar form and we can deduce* (iii) *and* (iv) *from it by simply using the "laws of indices"; (that is all de Moivre's theorem really is).*

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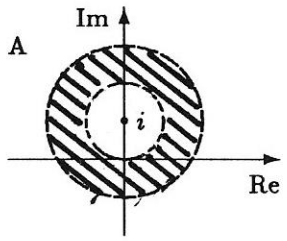
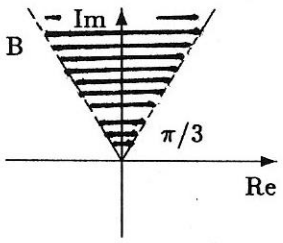
Question 2

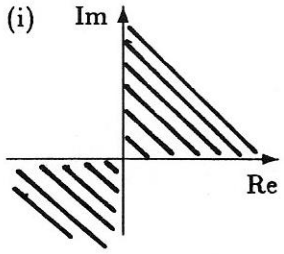


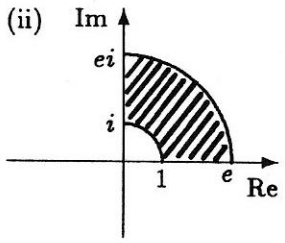
Connected
Not simply-connected
Region



Connected
Simply-connected
Region

Question 3





Comment *Note carefully whether it is the "image" or the "inverse image" that is required. In* (i) *the function is not one-one so you should expect more than one "bit" of the inverse image. It is helpful to think geometrically of what the function does.*

Part (ii) *is harder, but you should be aware that the modulus of* e^z *is* e^x *and its argument is* y*, where* $z = x + iy$*. So, as* x *varies from* 0 *to* 1 *the modulus of* e^z *(i.e. its distance from* 0*) varies from* e^0 *to* e^1 *and as* y *varies from* 0 *to* $\pi/2$ *the argument of* e^z *varies from* 0 *to* $\pi/2$*.*

Question 4

(i) (a) $i\pi/2$

(b) $1 + i = \sqrt{2}e^{i\pi/4}$
so $\mathrm{Log}(1+i) = \log\sqrt{2} + i\pi/4$.

(ii) $i^i = \exp(i \log i)$
$= \exp(i(i\pi/2 + i2k\pi)), \quad k \in \mathbf{Z}$
$= \exp(-(1/2 + 2k)\pi), \quad k \in \mathbf{Z}.$
Principal Value is $e^{-\pi/2}$.

(iii) Domain $= \{z : 1 - iz \neq -k, k \text{ real and non-negative}\}$
$= \{z : z \neq -i(k+1), k \text{ real and non-negative}\}.$

Comment *In (ii) you must specify* k *clearly.* $k = 0, 1, 2, \ldots$*, for example, missing out negative values, will lose 1/2 mark.*

Question 5

(i)
$$\int_\gamma \operatorname{Im} z\, dz = \int_0^1 \operatorname{Im}(\gamma(t)) \cdot \gamma'(t)\, dt$$
$$= \int_0^1 t \cdot (i-1)\, dt$$
$$= \frac{1}{2}(i-1).$$

Comment *In general* $\operatorname{Im} \int f \neq \int \operatorname{Im} f$. *Also* $\operatorname{Im} z$ *is not analytic so the Fundamental theorem does not apply. This is not hard. Follow the definition (Unit 4, page 30) step by step: put in the limits, work out* $f(\gamma(t))$, $\operatorname{Im}[1 + t(i-1)]$ *here, and find* $\gamma'(t)$.

(ii) (a) Use the parametrization $z = iy$ on $[i, 0]$
$$\int_{[i,0]} \operatorname{Im} z\, dz = \int_i^0 y \cdot i\, dy = -\frac{1}{2}i.$$

(b) As $\operatorname{Im} z = 0$ on the real axis $\int_{[0,1]} \operatorname{Im} z\, dz = 0$.
$$\int_\Gamma \operatorname{Im} z\, dz = \frac{1}{2}(i-1) - \frac{1}{2}i + 0 = -\frac{1}{2}.$$

Comment *You may choose a different parametrization for* (a) *e.g.* $z = (1-t)i$, $t \in [0,1]$, *but note that the one in* (i) *is unsuitable. In* (b) $z = x$, $x \in [0,1]$ *would be a suitable parametrization but then* $\operatorname{Im} x = 0$. *Finally note that as* Im *is not analytic you should expect the integral round the triangle to be nonzero.*

Question 6

(i) If f is analytic on a region R containing a circle C and its inside then for all z inside C, f is n times differentiable at z and
$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.$$
[The more general form given on page 98 of Unit 9 is equally acceptable.]

Comment *One mark for the formula and one for the conditions under which the formula holds. e.g. it must be clear that there cannot be singularities in or on* C *and that the formula applies only inside* C.

(ii) (a) As sin is analytic everywhere and π is inside C Cauchy's Formula for the first derivative is applicable.
$$\text{Integral} = \frac{2\pi i}{1} \sin'(\pi) = -2\pi i.$$

(b) $\dfrac{\sin z}{(z+\pi)^2}$ is analytic in and on C so Cauchy's Theorem applies and the integral is 0.

Comment *In both* (a) *and* (b) *half the marks are for checking that the theorems are applicable.*

Question 7

(i) $f(z) = \dfrac{-1/2}{z-1} + \dfrac{1/2}{z-3}$

(ii) The series is about 0 and as $1 < |z| < 3$ we must use $\dfrac{1}{z}$ for the first fraction and $\dfrac{z}{3}$ for the second (to be convergent as $\left|\dfrac{1}{z}\right| < 1$ and $\left|\dfrac{z}{3}\right| < 1$).

$$\begin{aligned} f(z) &= -\frac{1}{2}\cdot\frac{1}{z}\cdot\frac{1}{(1-1/z)} + \frac{1}{2}\cdot\left(-\frac{1}{3}\right)\cdot\frac{1}{(1-z/3)} \\ &= -\frac{1}{2z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\ldots\right) - \frac{1}{6}\left(1+\frac{z}{3}+\frac{z^2}{9}+\ldots\right) \\ &= \ldots - \frac{1}{2}z^{-2} - \frac{1}{2}z^{-1} - \frac{1}{6} - \frac{z}{18} - \ldots. \end{aligned}$$

Question 8

(i) (a) pole of order 3, (b) simple pole, (c) essential singularity.

Comment *Try to get the feel of the behaviour.*
e.g. in (a),
$\sin z = z - z^3/3! + \ldots$ *i.e.* $\sin z \sim z$ *(behaves like* z *near* 0*)*
so $\dfrac{\sin z}{z^4} \sim \dfrac{1}{z^3}$, *which has a pole of order 3. In* (c) $\dfrac{1}{z-1}$ *at* 1 *is like* $\dfrac{1}{z}$ *at* 0 *and since* $\sin\dfrac{1}{z}$ *has an essential singularity at* 0 *this function does at* 1. *Notice that in* (b) *the top is not* 0 *at* π *so we cannot have a removable singularity.* sin *at* π *behaves like* $-\sin$ *at* 0 *i.e. it has a zero of order* 1 *and hence* g *has a pole of order* 1 *at* π.

(ii) (a) $\dfrac{\sin z}{z^4} = \dfrac{1}{z^4}\left(z - \dfrac{z^3}{3!} + \ldots\right) = \dfrac{1}{z^3} - \dfrac{1}{3!z} + \ldots$

so $\operatorname{Res}(f, 0) = -1/6$.

(b) $\operatorname{Res}(g, \pi) = \dfrac{\pi}{\sin'\pi} = -\pi$. (Theorem 2 of Unit 10)

(c) $h(z) = (z-1)\left(\dfrac{1}{(z-1)} - \dfrac{1}{3!(z-1)^3} + \ldots\right)$

As there is no term in $\dfrac{1}{(z-1)}$ $\operatorname{Res}(h, 1) = 0$.

Comment *Working is needed in* (ii) *especially to justify a result of* 0. *Notice that for poles of higher orders a series method is usually easier than the derivative method.*

Question 9

The integrand is analytic except at 0, $\pm 2i$ which are all inside C. Therefore by the Residue Theorem

$$\begin{aligned}\text{Int} &= 2\pi i\Big[\,\text{Res}(0) + \text{Res}(2i) + \text{Res}(-2i)\Big] \\ &= 2\pi i\left[\frac{1}{4} + \frac{e^{2i}}{-8} + \frac{e^{-2i}}{-8}\right] \\ &= \frac{\pi i}{2}(1 - \cos 2).\end{aligned}$$

Comment *When you require residues at several simple poles the derivative method is often quicker than the cover-up rule. Here*

$$\text{Res}\left(\frac{e^z}{z(z^2+4)}, a\right) = \frac{e^a}{3a^2+4}.$$

Then give a the values 0, $2i$ and $-2i$.

For full marks you would be expected to simplify to the last line.

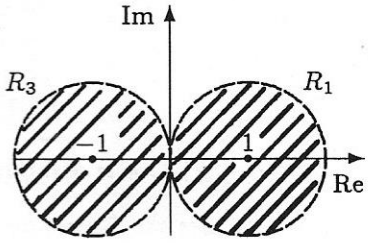
Question 10

(i) $f_1 = f_2$ on a nonempty open subset of $R_1 \cap R_2$.

(ii) $f_1(z) = \dfrac{1}{1-(1-z)} = \dfrac{1}{z}$ on R_1,

$f_3(z) = \dfrac{-1}{1-(1+z)} = \dfrac{1}{z}$ on R_3.

Define $f_2(z) = \dfrac{1}{z}$ on $R_2 = \mathbf{C} - \{0\}$.



Then (f_2, R_2) is a function element and $f_1 = f_2$ on $R_1 \cap R_2$, $f_1 = f_3$ on $R_1 \cap R_3$, and both $R_1 \cap R_2$ and $R_1 \cap R_3$ are nonempty open subsets of themselves.

So $\{(f_1, R_1), (f_2, R_2), (f_3, R_3)\}$ is a chain. Hence f_1 and f_3 are branches of the same complete analytic function.

Comment *Part* (ii) *is tough. You don't need to be able to do questions like this to pass M332 but you should be able to make a go of it if you are looking for a distinction. Other (f_2, R_2) will work, e.g. you might take R_2 to be $\{z : |i - z| < 1\}$ but then it is harder to prove $R_2 \cap R_1$ is a nonempty open set, or find a nonempty open subset of it.*

Question 11

(i) (a) $|z+w| \le |z| + |w|, \qquad |z-w| \ge ||z| - |w||.$

(b) If Γ is a contour and f continuous on Γ then $\left|\int_\Gamma f\right| \le ML$ where L is the length of Γ and M is an upper bound for $|f|$ on Γ.

Comment *Note that the upper bound is for $|f|$ not f and that it is on Γ. Overkill conditions such as "f analytic on a region containing Γ..." lose marks; all that is needed is that f is continuous on Γ — the theorem is about f on Γ only.*

We can assume that Γ is the directed path of a simple smooth arc γ (else Γ is the join of several such arcs). Let the domain of γ be $[a, b]$.

Then $\int_\Gamma f = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$

* Assuming $\left|\int_a^b \phi(t)\,dt\right| \le \int_a^b |\phi(t)|\,dt$ for ϕ complex on $[a, b]$,

$$\begin{aligned}\left|\int_\Gamma f\right| &\le \int_a^b |f(\gamma(t))||\gamma'(t)|\,dt \\ &\le M\int_a^b |\gamma'(t)|\,dt \\ &= ML.\end{aligned}$$

Comment *A vague $|\int f| \le \int |f|$ at * is insufficient.*

(ii) *Comment* *The strategy is to use the Estimation Theorem so we must check that the conditions hold and find a suitable M and L. Expect to use*(i)(a) *as well.*

Γ is a semicircle of radius 1, so $L = \pi$.

To find M we need an upper bound for $|z + \operatorname{Log} z|$ and a lower positive bound for the bottom $|e^z + 3|$ on Γ.

$$\begin{aligned}|z + \operatorname{Log} z| &\le |z| + |\operatorname{Log} z| \\ &= 1 + |\log|z| + i\operatorname{Arg} z| \\ &\le 1 + |\log 1| + |\theta| \quad \text{on } \Gamma \\ &\le 1 + 0 + \pi/2 = 1 + \pi/2.\end{aligned}$$

$$\begin{aligned}|e^z + 3| &= |e^{\cos\theta + i\sin\theta} + 3| \quad \text{on } \Gamma \\ &\ge |e^{\cos\theta} - 3| \\ &= 3 - e^{\cos\theta} \\ &\ge 3 - e.\end{aligned}$$

Notice that $3 - e > 0$ and so $e^z + 3 \ne 0$ on Γ. This together with the fact that Γ lies inside the region $\{z : z \ne -k$ where k real and nonnegative$\}$ on which Log is analytic means that the function is continuous on Γ — so the Theorem applies.

Put $M = \dfrac{1 + \pi/2}{3 - e}$.

Hence $\left|\int_\Gamma f\right| \le \dfrac{1 + \pi/2}{3 - e} \cdot \pi.$

Question 12

(i) (a) $\dfrac{\partial U}{\partial x}(x_0, y_0) = \dfrac{\partial V}{\partial y}(x_0, y_0)$ and $\dfrac{\partial U}{\partial y}(x_0, y_0) = -\dfrac{\partial V}{\partial x}(x_0, y_0)$.

Comment *Omission of (x_0, y_0) loses a mark. The equations need only hold at the point, not in general.*

(b) $U(x, y) = x^2 + y^2 - y$, $V(x, y) = x$.

$\dfrac{\partial U}{\partial x} = 2x$, $\dfrac{\partial U}{\partial y} = 2y - 1$, $\dfrac{\partial V}{\partial x} = 1$ and $\dfrac{\partial V}{\partial y} = 0$.

At 0 these are 0, -1, 1, 0 so Cauchy-Riemann equations hold at 0.
At $1 + i$ these are 2, 1, 1, 0 so Cauchy-Riemann equations fail at $1 + i$.

(c) As Cauchy-Riemann equations fail at $1 + i$ the function f is not differentiable there. (Reason must be stated.)
At 0 the Cauchy-Riemann equations hold AND the partial derivatives are continuous there so f is differentiable at 0. (See Theorem 15 of Unit 3.)

$$f'(0) = \frac{\partial U}{\partial x}(0, 0) + i\frac{\partial V}{\partial x}(0, 0) = i.$$

(ii) (a) f is differentiable at a point z_0 if and only if $\lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists.

$$\frac{g(z) - g(z_0)}{z - z_0} = \frac{\operatorname{Re} z - \operatorname{Re} z_0}{z - z_0} = \frac{\operatorname{Re}(z - z_0)}{z - z_0}$$

In particular this is 1 if $z - z_0$ is real and is 0 if $z - z_0$ purely imaginary.

So $\lim\limits_{z \to z_0} \dfrac{g(z) - g(z_0)}{z - z_0}$ cannot exist for any z_0,

i.e. g is not differentiable anywhere.

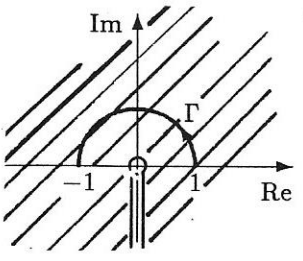
(b) False.
e.g. if $h_1 : z \longrightarrow \operatorname{Re} z$ and $h_2 : z \longrightarrow \operatorname{Im} z$
then $h : z \longrightarrow z$ which is differentiable everywhere.

Question 13

(i) Fundamental Theorem. There is a branch L_R of the logarithm analytic on the shaded region.

So $\displaystyle\int_\Gamma \frac{1}{z}\,dz = \Big[L_R(z)\Big]_1^{-1} = i\pi - 0 = i\pi.$

[Or use parametrization $z = e^{i\theta}$, $\theta \in [0, \pi]$.]



(ii) Let $z = e^{i\theta}$, $\theta \in [0, \pi]$.

$$\int_\Gamma |z|\,dz = \int_0^\pi 1 \cdot ie^{i\theta}\,d\theta = e^{i\pi} - 1 = -2.$$

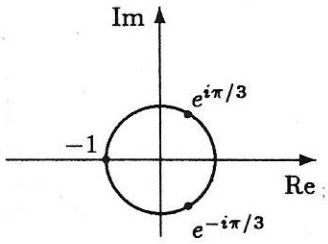
Comment *$|z|$ is not analytic anywhere so the parametrization method is mandatory. $e^{i\pi}$ must be simplified.*

(iii) $\dfrac{z}{\sin z}$ is analytic within a region containing C except at 0, so by the Residue Theorem

$$\int_C \frac{z}{\sin z}\,dz = 2\pi i \operatorname{Res}(0) = 0$$

as the singularity at 0 is removable.

(iv) $\dfrac{1}{1+z^3}$ is analytic in C except when $z^3 = -1$, so by the Residue Theorem



$$\begin{aligned}
\int_C \frac{1}{1+z^3}\,dz &= 2\pi i\left[\operatorname{Res}(-1) + \operatorname{Res}(e^{i\pi/3}) + \operatorname{Res}(e^{-i\pi/3})\right] \\
&= 2\pi i\left[\frac{1}{3(-1)^2} + \frac{1}{3(e^{i\pi/3})^2} + \frac{1}{3(e^{-i\pi/3})^2}\right] \\
&= \frac{2\pi i}{3}\left[1 + e^{-i2\pi/3} + e^{i2\pi/3}\right] \\
&= \frac{2\pi i}{3}\left[1 + \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3} + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right] \\
&= 0.
\end{aligned}$$

Comment *As in Question 9 it is quicker to use the derivative method to find the residues than the cover-up rule and as in Question 1 much easier to use the exponential form of the points where the singularities occur than the $a+ib$ form: compare using the index law to write $\dfrac{1}{(e^{i\pi/3})^2}$ as $e^{-i2\pi/3}$ to evaluating $\dfrac{1}{\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2}$.*

(v) cosh is analytic everywhere and $\log 2$ is inside C. Hence, using Cauchy's Formula for the 5th derivative

$$\begin{aligned}
\int_C \frac{\cosh z}{(z-\log 2)^6}\,dz &= \frac{2\pi i}{5!}\cosh^{(5)}(\log 2) \\
&= \frac{\pi i}{60}\sinh(\log 2) \\
&= \frac{\pi i}{60}\cdot\frac{1}{2}\left[e^{\log 2} - e^{-\log 2}\right] \\
&= \frac{\pi i}{120}\left[2 - \frac{1}{2}\right] = \frac{\pi i}{80}.
\end{aligned}$$

Comment $\sinh(\log 2)$ *must be simplified completely for full marks.*

Question 14

(i) (a) If f is analytic on a bounded region R and continuous on $\overline{R}$ then $|f|$ has a maximum on $\overline{R}$ at a point on the boundary of R.

(b) The Maximum Principle applies to $f(z) = \dfrac{z^2-3}{z^2+3}$ in $|z|<1$ as $z^2 \neq -3$ in $|z| \leq 1$. So $f(z)$ has a maximum on $|z|=1$. On $|z|=1$, $z = e^{i\theta}$ so

$$\begin{aligned}|f(z)| &= \left|\frac{(e^{i\theta})^2-3}{(e^{i\theta})^2+3}\right| = \left|\frac{e^{i2\theta}-3}{e^{i2\theta}+3}\right| \\ &= \left|\frac{(\cos 2\theta - 3)+i\sin 2\theta}{(\cos 2\theta+3)+i\sin 2\theta}\right| \\ &= \sqrt{\frac{(\cos 2\theta-3)^2+\sin^2 2\theta}{(\cos 2\theta+3)^2+\sin^2 2\theta}} \\ &= \sqrt{\frac{10-6\cos 2\theta}{10+6\cos 2\theta}}.\end{aligned}$$

This has a maximum when $\cos 2\theta = -1$ and is $\sqrt{\frac{16}{4}} = 2$, and occurs at the points when $\theta = \pi/2$ or $3\pi/2$ i.e. at $\pm i$.

Comment *It is not sufficient to find a maximum for $|z^2-3|$ and a minimum for $|z^2+3|$ on $|z|=1$ and divide the first by the second unless (i) they both occur at the same point, and (ii) you can be sure that you have obtained all the points where the maximum of $|f(z)|$ can be attained. Other methods do work but this one is fairly general in its application.*

(ii) (a) As sin is entire the Maximum Principle applies to sin on D_a. Thus $|\sin|$ attains a maximum value in $\overline{D}_a$ which gives an upper bound for $|\sin|$ on D_a. Thus sin is bounded on D_a.

(b) As sin is entire, if it were bounded on $\mathbf{C}$ it would, by Liouville's Theorem, be a constant function, which it is not. Hence it is not bounded on $\mathbf{C}$.

(c) Cover $\mathbf{C}$ with open discs $\{D_a : a \in \mathbf{C}\}$. If $\mathbf{C}$ is compact then it can be covered by a finite number of them and $|\sin|$ would be bounded by the largest of its upper bound on these particular D_a, contrary to (b). Thus $\mathbf{C}$ is not compact.

Question 15

(i) (a) [Theorem 8b of Unit 10] $p(t) = t$, $q(t) = t^2 + 1$ are polynomials, q has no zeros on the real axis and the degree of q is one greater than that of p, and $\beta = 2$ is greater than zero.

So $\displaystyle\int_{-\infty}^{\infty} \frac{t \sin 2t}{t^2+1}\, dt = \mathrm{Im}(2\pi i\, S + \pi i\, T)$

where S is the sum of residues of $\dfrac{ze^{2iz}}{z^2+1}$ in upper half-plane and T is the sum of residues on the real axis.

$T = 0$ and $S = \mathrm{Res}(i) = e^{-2}/2$.
So the integral is $\mathrm{Im}(2\pi i e^{-2}/2) = \pi e^{-2}$.

Comment *Theorem 7 (Unit 10) is easier to use here but Theorem 8 subsumes it and so we need not remember the earlier theorem.*

(b) As $\dfrac{1}{t^2-4}$ is an even function $\displaystyle\int_0^{\infty} \frac{1}{t^2-4}\, dt = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{t^2-4}\, dt$.

[Theorem 8a] $p(t) = 1$, $q(t) = t^2 - 4$ are polynomials with the degree of q two more than that of p, q has simple zeros at ± 2 on the real axis and $\beta = 0$.

So $\displaystyle\int_{-\infty}^{\infty} \frac{1}{t^2-4}\, dt = \mathrm{Re}(2\pi i\, S + \pi i\, T), \quad (1 = 1 \cdot \cos 0t)$

where S and T are as above for the function $\dfrac{1e^{0z}}{z^2-4} = \dfrac{1}{z^2-4}$.

Here $S = 0$ and $T = \mathrm{Res}(-2) + \mathrm{Res}(2) = -\frac{1}{4} + \frac{1}{4} = 0$.

So $\displaystyle\int_0^{\infty} \frac{1}{t^2-4}\, dt = \frac{1}{2}\,\mathrm{Re}(0) = 0$.

(ii) Let $z = e^{i\theta}$, $\theta \in [-\pi, \pi]$, then $\cos\theta = \frac{1}{2}(z + 1/z)$ and $d\theta = \dfrac{dz}{iz}$.

So $\displaystyle\int_{-\pi}^{\pi} \frac{1}{5 + 4\cos\theta} d\theta = \int_C \frac{1}{5 + \frac{4}{2}(z + 1/z)} \cdot \frac{dz}{iz}$,

and so $f(z) = \dfrac{-i}{2z^2 + 5z + 2}$.

Integral $= \displaystyle\int_C \frac{-i}{(2z+1)(z+2)}\, dz$. *

The function f is analytic except at -2 and $-1/2$ so Residue Theorem applies with the only singularity inside C being at $-1/2$,

$\mathrm{Res}(f, -1/2) = \dfrac{-i}{4(-1/2) + 5} = -\dfrac{i}{3}$. [Differentiation method]

So Int $= 2\pi i(-i/3) = 2\pi/3$.

Comment *All these integrals are real integrals so expect real answers!*

*Be careful if you use the cover-up rule in situations such as *. You can only cover-up a bracket of the form $(z \pm \ldots)$ so* $\dfrac{-i}{(2z+1)(z+2)}$ *would have to be written* $\dfrac{-i}{2(z+1/2)(z+2)}$ *before using the rule.*

Question 16

(i) If f is meromorphic on a simply-connected bounded region R, Γ is a simple-closed contour in R with f having no zeros or poles on Γ, then

$$N - P = \frac{1}{2\pi i}\int_\Gamma f'/f.$$

Comment *A marker would not be concerned that you had not defined N or P but the conditions of the theorem are very important. e.g. consider what would go wrong if Γ were not simple-closed or R not simply-connected.*

(ii) If f and g are analytic on a simply-connected region R, Γ is a simple-closed contour in R and if $|f(z)| > |g(z)|$ on Γ then f and $f+g$ have the same number of zeros inside Γ (counting according to order).

Comment *The facts that we require are that $|f(z)| > |g(z)|$ <u>on Γ</u> and that the zeros considered are <u>inside Γ</u> are crucial to the theorem. No mention of Γ gets zero marks.*

$$\begin{aligned} N(f+g) - N(f) &= \frac{1}{2\pi i}\int_\Gamma \frac{f'+g'}{f+g} - \frac{1}{2\pi i}\int_\Gamma \frac{f'}{f} \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{fg' - f'g}{f(f+g)} \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{\phi'}{1+\phi}, \text{ where } \phi = \frac{g}{f}. \end{aligned}$$

As $|f(z)| > |g(z)| \geq 0$ on Γ, $f \neq 0$ on Γ and so ϕ is defined and analytic on Γ. Hence by the assumption, as $|\phi| = \frac{|g|}{|f|} < 1$ on Γ,

$$N(f+g) - N(f) = 0 \text{ i.e. } N(f+g) = N(f).$$

Comment *Checking that the conditions of the assumption hold is required for full marks.*

(iii) Γ is $|z| = 2$. Let $f(z) = 6z^2 + 2$, $g(z) = e^{iz}$. [$6z^2$ and $e^{iz} + 2$ also work.]

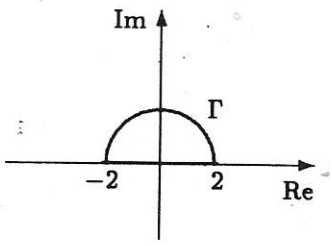
$|g(z)| = |e^{iz}| = e^{\mathrm{Re}(iz)} = e^{-\mathrm{Im}\, z} \leq e^2 < 9$ on $|z| = 2$.
$|f(z)| = |6z^2 + 2| \geq ||6z^2| - 2| \geq 24 - 2 = 22$ on $|z| = 2$.

Hence $|f(z)| > |g(z)|$ on $|z| = 2$ and so f and $h = f + g$ have the same number of zeros within $|z| = 2$ (by Rouché). Now the zeros of $f(z)$ are $\pm i\sqrt{1/3}$ both inside $|z| = 2$, and so h also has two zeros inside $|z| = 2$.

Comment *This is not hard but requires small steps and careful working. Notice that we require to show $|f| > |g|$ on Γ so we use the cunning form of the triangle inequality on $|f|$ to show it bigger that $|g|$. Sometimes we use the normal form on $|g|$ or both. Also note that we must check that the zeros of f are inside Γ.*

The last part requires more thought. See also SAQ 2 on page 103 of Unit 11.

Using $f(z) = 6z^2 + 2$ and $g(z) = e^{iz}$ as above we know that $|f(z)| > |g(z)|$ on the semicircular part of Γ shown in the figure.



On $[-2, 2]$, $|g(z)| = |e^{-\mathrm{Im}\, z}| = e^0 = 1$
and $|f(z)| = 6x^2 + 2 \geq 2$. So $|f| > |g|$ on $[-2, 2]$.

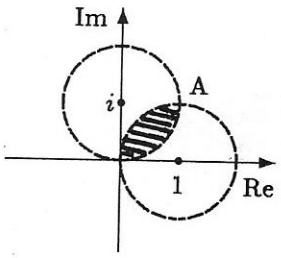
Thus $|f(z)| > |g(z)|$ on Γ and so by Rouché f and $h = f + g$ have the same number of zeros inside Γ, namely 1 (f has one at $i\sqrt{1/3}$). So exactly one of the zeros of h is in the upper half-plane.

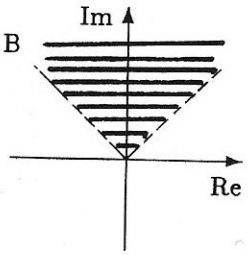
Question 17

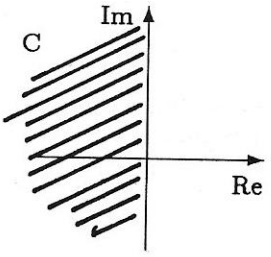
(i) A conformal mapping is one which preserves angles between arcs.
A bilinear transformation is one of the form

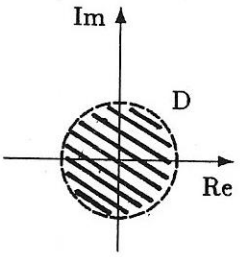
$$z \longrightarrow \frac{az+b}{cz+d}, \qquad ad \neq bc.$$

(ii) (a)





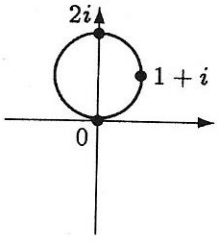


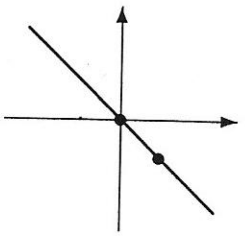


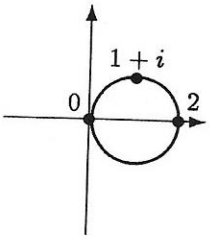
(b) line/circle $\longrightarrow$ line/circle
$0 \longrightarrow \infty$ (so line)
$1+i \longrightarrow 0$
$2i \longrightarrow \dfrac{1-i}{2}$ so image is line "$y=-x$".

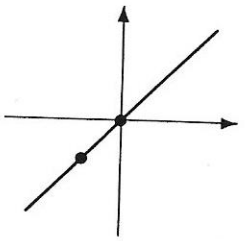
As above $0 \longrightarrow \infty$ and $1+i \longrightarrow 0$
$2 \longrightarrow \dfrac{1-i}{2i} = -\dfrac{1+i}{2}$
so image is line "$y=x$".

So boundaries of $A \longrightarrow$ boundaries of B.
We still have to check that region A maps to the correct area bounded by the lines.
$\dfrac{(1+i)}{2}$ is inside A and $\dfrac{1+i}{2} \longrightarrow -\dfrac{1}{i} = i \in B$.









(c) $g : z \longrightarrow z^2$

(d) *Comment* *There are many ways of doing this e.g. if you remember that* $z \longrightarrow \frac{1+iz}{1-iz}$ *maps the upper half-plane into the unit circle you need only rotate* C *through* $-\pi/2$ *first using* $z \longrightarrow -iz$ *and compose with the above map, or you might notice that* $z \in C$ *iff* z *is nearer* -1 *than* 1. *i.e.* $|z+1| < |z-1|$ *or* $|\frac{z+1}{z-1}| < 1$
i.e. $z \in C \iff \frac{z+1}{z-1} \in D$. *The following is a slower but more generally applicable method.*

Let $h : z \longrightarrow \dfrac{az+b}{cz+d}$ and take (e.g.)

$$0 \longrightarrow 1, \quad \text{so } b/d = 1 \text{ or } d = b,$$
$$\text{and} \quad \infty \longrightarrow -1, \quad \text{so } -a/c = 1 \text{ or } c = -a.$$

Thus $h : z \longrightarrow \dfrac{z+(b/a)}{-z+(b/a)}$.

Let $i \longrightarrow i$ so that $\dfrac{i+(b/a)}{-i+(b/a)} = i$ giving $(b/a) = 1$.

Thus $h : z \longrightarrow \dfrac{1+z}{1-z}$ takes the boundary of C to that of D
$-1 \longrightarrow 0$ so $C \longrightarrow D$ so this h is suitable.

Comment *If your* h *mapped* C *to the outside of the unit circle you would simply take its reciprocal.*

Note that there are an infinite number of correct answers here corresponding to different transformations of C *to itself e.g. rotations.*

(e) The mapping we require is

$$h \circ g \circ f : z \longrightarrow \frac{1 + \left(\frac{z-(1+i)}{iz}\right)^2}{1 - \left(\frac{z-(1+i)}{iz}\right)^2}.$$

Each of f, g, h is analytic, one-one (on the region concerned) and onto, so $h \circ g \circ f$ is too.

Question 18

(i) $F(p) = \int_0^\infty f(t)e^{-pt}\,dt$

(a)
$$\begin{aligned}\mathcal{L}[f'](p) &= \int_0^\infty f'(t)e^{-pt}\,dt \\ &= \left[f(t)e^{-pt}\right]_0^\infty - \int_0^\infty f(t)(-pe^{-pt})\,dt \\ &= 0 - f(0) + pF(p) \qquad \text{as } f(t)e^{-pt} \longrightarrow 0 \text{ as } t \longrightarrow \infty \\ &= -f(0) + pF(p).\end{aligned}$$

Applying this to f'

$$\begin{aligned}\mathcal{L}[f''](p) &= -f'(0) + p\mathcal{L}[f'](p) \\ &= -f'(0) + p(-f(0) + pF(p)) \\ &= -f'(0) - pf(0) + p^2F(p).\end{aligned}$$

(b)
$$\begin{aligned}\mathcal{L}[\sin](p) &= \int_0^\infty \sin t\, e^{-pt}\,dt \\ &= \frac{1}{2i}\int_0^\infty (e^{it} - e^{-it})e^{-pt}\,dt \\ &= \frac{1}{2i}\left[\frac{e^{(i-p)t}}{i-p} - \frac{e^{-(i+p)t}}{-(i+p)}\right]_0^\infty \\ &= \frac{1}{2i}\left[-\left(\frac{1}{i-p} + \frac{1}{i+p}\right)\right] = \frac{1}{p^2+1}.\end{aligned}$$

Comment *Alternatively use integration by parts but writing* sin *in exponential form is quicker.*

(ii) (a) $f''(t) + f(t) = \sin t$ transforms to $p^2F(p) + F(p) = \dfrac{1}{p^2+1}$,

so $F(p) = \dfrac{1}{(p^2+1)^2}$.

(b) (1) F is analytic on $\operatorname{Re} p \geq x_0$ for any $x_0 > 0$ e.g. $x_0 = 1$.

(2) $|p^2+1| \geqslant |p|^2 - 1 \geqslant |p|^2/2$ if $|p|^2 \geq 2$, so take p_0 any number $\geq \sqrt{2}$, say 2. Then $|F(p)| \leq 4|p|^{-4}$ for $|p| \geq p_0 = 2$.

(3) When p is real $F(p)$ is real.

So we may use inversion formula.

(c) $f(t)$ = sum of the residues of $\dfrac{e^{pt}}{(p^2+1)^2}$.

Singularities are at $\pm i$ and are poles of order 2.

$$\begin{aligned}\text{Res}(i) &= \lim_{p\to i}\frac{d}{dp}\left(\frac{e^{pt}}{(p+i)^2}\right) \\ &= \lim_{p\to i}\left(\frac{te^{pt}}{(p+i)^2} - \frac{2e^{pt}}{(p+i)^3}\right) \\ &= -\frac{1}{4}te^{it} - \frac{1}{4}ie^{it}.\end{aligned}$$

$$\begin{aligned}
\text{Res}(-i) &= \lim_{p\to -i}\left(\frac{te^{pt}}{(p-i)^2} - \frac{2e^{pt}}{(p-i)^3}\right)\\
&= -\frac{1}{4}te^{-it} + \frac{1}{4}ie^{-it}.\\
\text{So } f(t) &= -\frac{1}{4}t(e^{it}+e^{-it}) - \frac{i}{4}(e^{it}-e^{-it})\\
&= -\frac{t}{2}\cos t + \frac{1}{2}\sin t.
\end{aligned}$$

Comment *Do the working-out, and if it is different from the given answer and you cannot find your error leave your answer. You will lose more marks if you simply copy the answer from the question paper when it does not follow from your working. It is easy to make a small slip working out the residues and if your final answer is consistent with it you may lose only one mark.*

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